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OPTIMAL ORBITAL REGULATION IN DYNAMICAL SYSTEMS SUBJECT TO HOPF--ETC(U)
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ABSTRACT

We consider a controlled system

$$\dot{x} = f(x,u)$$

exhibiting a self excited periodic solution $x(t)$ for $u(t) \equiv 0$ and consider the question of modification of this orbit to a controlled periodic solution $\hat{x}(t)$ corresponding to a control $\hat{u}(t)$ chosen so as to minimize a cost functional of the form

$$\frac{1}{\hat{T}} \int_0^{\hat{T}} w(\hat{x}(t), \hat{u}(t))dt \quad ,$$

where $\hat{T}$ is the period of the controlled periodic solution. Some relevant applications are cited.

AMS (MOS) Subject Classifications: 93C15, 49A10, 49B10, 34C25

Key Words: Periodic Systems, Periodic Control, Optimal Control

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SIGNIFICANCE AND EXPLANATION

Great pains have been taken to make Section 1 of this paper an explanation of the background and significance of the work. This is particularly true of the first three pages of the section. This material requires minimal technical background and is recommended to the reader wishing to appreciate the basic ideas of the paper without getting too involved in the technical material.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

OPTIMAL ORBITAL REGULATION IN DYNAMICAL SYSTEMS

SUBJECT TO HOPF BIFURCATION*

David L. Russell**

1. Introduction. Many systems involving self-excited, nonlinear oscillations can be usefully modelled by a system of the form

$$\dot{x} = A(\theta)x + B(\theta)\binom{y}{\dot{y}} + D(\theta)u \quad , \tag{1.1}$$

$$x \in E^n, \ u \in E^m \quad ,$$

$$\ddot{y} + \gamma(\theta,\dot{y})\dot{y} + k(\theta)y = c(\theta)^* x + e(\theta)^* u \quad , \tag{1.2}$$

$$y \ \text{scalar.}$$

The first system (1.1) in which $A(\theta)$, $B(\theta)$, $D(\theta)$ are matrices of dimension $n \times n$, $n \times 2$, $n \times m$, respectively, represents a linear oscillator of some sort with a degree of internal damping, i.e., the eigenvalues of $A(\theta)$ have negative real parts for all values of the parameter θ under consideration. The scalar second order equation (1.2) in y represents a nonlinear oscillator. In it $c(\theta)$, $e(\theta)$ are vectors of dimension n, m, respectively, and * denotes transpose. In most cases the uncoupled equation

$$\ddot{y} + \gamma(\theta,\dot{y})\dot{y} + k(\theta)y = 0 \tag{1.3}$$

has $y = \dot{y} = 0$ as an asymptotically stable critical point for θ in some range, say $0 \leq \theta < \theta_0$, that critical point becoming unstable for $\theta > \theta_0$. Typically the term $\gamma(\theta,\dot{y})\dot{y}$ takes the form

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$$\gamma(\theta,\dot y)\dot y = \gamma_0\dot y - \gamma_1(\theta)\dot y + \hat\gamma(\theta,\dot y)\dot y \quad ,$$

where $\gamma_0 > 0$, $\theta\gamma_1(\theta) > 0$, $\theta \neq 0$. The higher order terms represented by $\hat\gamma(\theta,\dot y)\dot y$ provide strong restoring forces when $\dot y$ is large. The familiar consequence, amply documented in the literature (see, e.g., [A], [B], [C]) is that for $\theta > \theta_0$ (1.3) has a stable periodic solution expanding rapidly away from the origin $(0,0)$ as θ increases beyond θ_0.

The uncontrolled coupled system ((1.1), (1.2) with $u \equiv 0$) typically involves a matrix $B(\theta)$ which is rather small, so that the matrix describing the linearization at the origin,

$$\begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta)^* & \begin{bmatrix} 0 & 1 \\ -k_0(\theta) & -\gamma_0 + \gamma_1(\theta) \end{bmatrix} \end{pmatrix} \tag{1.4}$$

is nearly lower block triangular and is a stability matrix for $\theta < \theta_1$, becoming unstable for $\theta > \theta_1$, where θ_1, the bifurcation point, is near the θ_0 value already discussed for (1.3). The system (1.1), (1.2) then likewise exhibits a stable periodic solution for $\theta > \theta_1$, lying close to the y, $\dot y$ plane for θ near θ_1.

One of the simplest examples is the Hartlen-Curry model ([D]) for aerodynamically induced flutter of a long elastic rod of circular cross section as shown schematically in Figure 1.1. A stream of fluid, e.g., air, flows past the rod with velocity V. For a certain "flutter speed", V_0, the fluid flow is steady for $V < V_0$. For $V > V_0$ the fluid flow is no longer steady; alternating vortices form in the wake behind the rod, producing a near-periodic force of alternating direction with a frequency, dependent on V, which is known as the Strouhal frequency. The Hartlen-Curry model uses equations of the form (cf. [E])

$$\ddot{x} + \beta\dot{x} + \rho x = aV^2 y$$

$$\ddot{y} + (\sigma - \alpha V)\dot{y} + \frac{\gamma}{V}\dot{y}^3 + \delta V^2 y = b\dot{x}$$

to describe the self-excited oscillations of the coupled system for $V > V_0$ (V_0 is slightly greater than σ/α). Typically β, ∂, σ, α, b are rather small quantities.

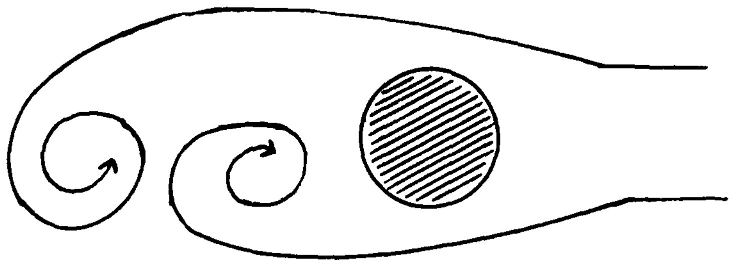

Figure 1.1 Oscillations of a rod with circular cross section.

A more complicated example, for which we will not write out the equations in detail, is of considerable interest in flight dynamics. It is rather similar to the example just presented for the rod with circular cross section, except that we now envision a wing cross-section immersed in the flowing fluid as shown in Figure 1.2. The displacements shown in the figure are, of course, exaggerated by comparison with actual operating levels (one hopes!).

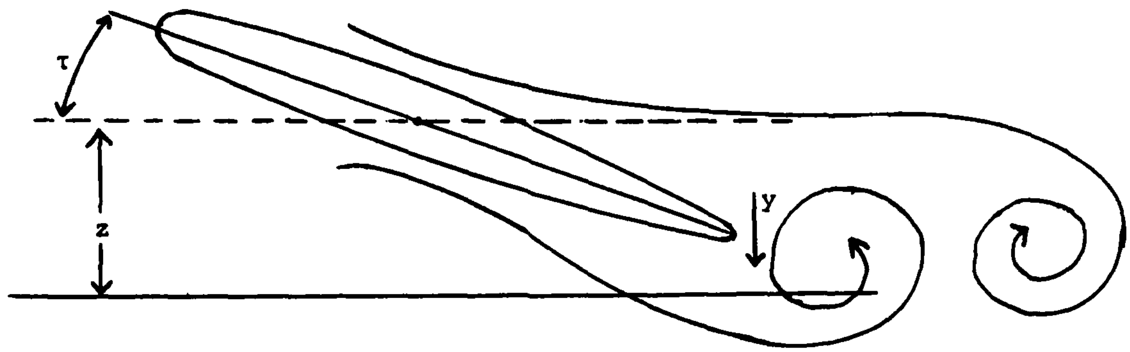

Figure 1.2

Here z represents the vertical displacement of the wing tip, τ is the torsional displacement of the wing at the tip, and, again, y is the vortex strength measured at the trailing edge of the wing. One obtains here a six dimensional system

$$\ddot{z} = f(z,\dot{z},\tau,\dot{\tau},y,\dot{y},u,V)$$
$$\ddot{\tau} = g(z,\dot{z},\tau,\dot{\tau},y,\dot{y},u,V)$$
$$\ddot{y} = h(z,\dot{z},\tau,\dot{\tau},y,\dot{y},u,V)$$

exhibiting self excited oscillations much as in the Hartlen-Curry case with added features, such as divergence of the periodic solution to infinity at certain parameter values for V.

The initial oscillations arising in systems of this sort are, at least insofar as the x component (cf. (1.1), (1.2)) is concerned, of rather small amplitude. The exception which causes greatest concern occurs when the nonlinear oscillator equation (1.3) has a periodic solution whose frequency is close to one of the natural frequencies of vibration of the linear elastic system modelled by $\dot{x} = A(\theta)x$. When this situation obtains, quite dramatic increases in amplitude may result in (1.1), (1.2). Figure 1.3 shows, qualitatively, the sort of results that one obtains with a Hartlen-Curry type model.

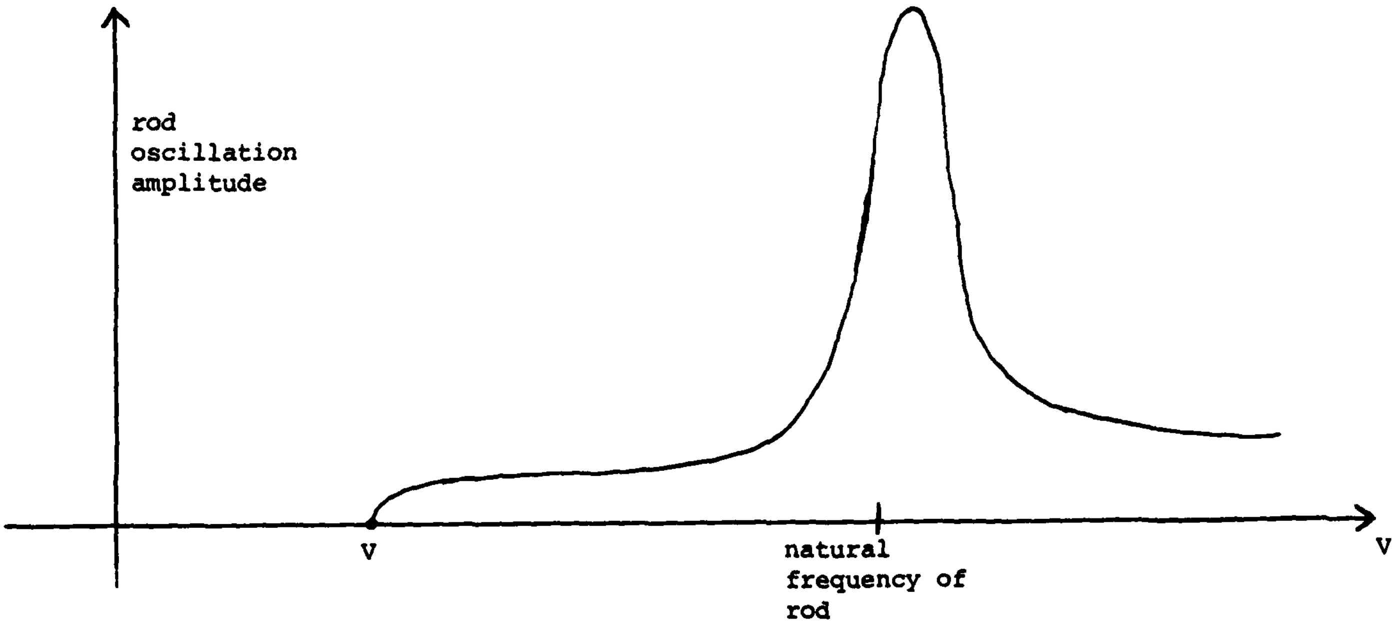

Figure 1.3

Among other things, control may be used in (1.1), (1.2) to increase the range of parameter values θ for which the origin is asymptotically stable, i.e., to increase θ_1 and/or to modify the nature of the oscillations which do occur for $\theta > \theta_1$ - by suppressing the amplitude of those oscillations, for example.

The traditional mode of control is linear feedback, the general form of which, for (1.1), (1.2) would be

$$u = Kx + L\binom{y}{\dot{y}} \quad , \tag{1.5}$$

possibly with gains scheduled so that $K = K(\theta)$, $L = L(\theta)$. It will be recognized, however, that direct measurement of y, $\dot{y}$ will often, perhaps usually, be impractical. Certainly the measurement of the vortex strength at the trailing edge of a wing would require a sensory <u>tour de force</u>. So, in practice, (1.5) must ordinarily be modified to

$$u = Kx \quad . \tag{1.6}$$

When one investigates how (1.6) affects (1.4), one sees that the result is to replace that matrix by

$$\begin{pmatrix} A(\theta) + D(\theta)K & B(\theta) \\ c(\theta)^* + e(\theta)^* K & \begin{bmatrix} 0 & 1 \\ -k_0(\theta) & -\gamma_0 + \gamma_1(\theta) \end{bmatrix} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \tilde{A} & B \\ C & \Gamma \end{pmatrix} . \tag{1.7}$$

One may verify that if T satisfies the quadratic matrix equation

$$\tilde{A}T - T\Gamma - TCT + B = 0 \tag{1.8}$$

then (1.7) is similar to

$$\begin{pmatrix} \tilde{A} - TC & 0 \\ C & CT + \Gamma \end{pmatrix} . \tag{1.9}$$

For $\theta > \theta_0$ Γ is unstable and it may be assumed that K is selected so that $\tilde{A}$ remains stable. It is then classical that

$$\tilde{A}T - T\Gamma + B = 0 \tag{1.10}$$

has a unique solution T and an easy application of the implicit function theorem shows that for small B (1.8) has a unique solution near the solution of (1.10). Moreover, as B tends to zero, T tends to zero. Now our capability to influence the eigenvalues of $CT + \Gamma$ lies entirely in the term CT appearing in (1.9). If B is quite small, large changes in K are required to produce modest changes in T. It follows that when B is small, as is commonly the case, it will be difficult to materially affect the eigenvalues which are responsible for the bifurcation phenomena, i.e., the eigenvalues of $CT + \Gamma$. This means that restricted feedback of the form (1.6) is not likely to be very effective in extending the range of values of θ for which the origin remains asymptotically stable.

It may be argued that feedback of the form (1.5) may be very nearly realized through the use of a state estimator ([F]) for $y, \dot{y}$. However, it is likely that the estimation will be very difficult in practice because, when B is near 0, $y, \dot{y}$ are nearly unobservable via measurements on x and, moreover, the oscillations of (1.2) necessarily take place in a region where the nonlinear terms appearing there balance the linear terms, "energy wise", and hence are of comparable magnitude. Then one is trying to construct a state estimator for nearly unobservable, nonlinear phenomena; a rather heroic, not to say Sisyphic, task.

It follows that if linear feedback is to be used, it will take the form (1.6) and the main benefit will be realized through choosing K so that the term $B(\mu)\binom{y}{\dot{y}}$ in (1.1) affects x minimally in some appropriate sense. For example, the eigenvalue of $A(\theta) + D(\theta)K$ corresponding to the fundamental mode of vibration can be moved further to the left in the complex plane or its imaginary part can be increased so that resonance occurs at higher frequencies, thus, in effect, pushing the "spike" of Figure 1.3 to the right. The main point, insofar as our current discussion is concerned, is that one must, in effect, concede that oscillation is going to take place and then take steps to modify or suppress the manifestations of such oscillations in the system represented by (1.1).

Having seen that we may as well admit from the first that a self-excited periodic motion will be present in the operating system, we may as well cite this as an excuse and an opportunity to deal directly with such solutions as we attempt regulation of the system (1.1), (1.2). Our purpose in this paper is to discuss the optimal regulation of periodic solutions of systems of this type, primarily with a view toward amplitude suppression.

In Sections 2, 3 we study a general system (wherein θ is suppressed until needed)

$$\dot{x} = f(x,u) \tag{1.11}$$

for $x \in O$, an open set in R^n, and $u \in E^m$. Most of our results are actually obtained in the context of the specialized systems of the form

$$\dot{x} = g(x) + H(x)u \quad .$$

Our objective is to show the existence of, and to characterize, in terms of necessary conditions, state and control pairs $\hat{x}$, $\hat{u}$ such that x is periodic with period $\hat{T}$ and such that $\hat{x}$, $\hat{u}$ minimize, relative to an appropriate family of state and control pairs x, u (x with period T), a cost functional of the general form

$$J(x,u,T) = \frac{1}{T}\int_0^T w(x(t),\, u(t))dt \quad . \tag{1.12}$$

In Section 2 we begin by supposing that the uncontrolled system

$$\dot{x} = f(x,0)$$

has a periodic solution with period T_0 and then obtain some local existence results ensuring the existence of an optimal control, at least in an appropriate local sense. Then in Section 3 we indicate the form of the necessary conditions, some computational methods and describe some computational work already undertaken, concluding with an indication of work in progress and envisioned for the future.

2. Local Existence of a Solution for the Periodic Optimal Control Problem

In our study of existence questions we will confine attention to systems in which the control appears linearly:

$$\dot{x} = f(x,u) = g(x) + H(x)u \quad , \tag{2.1}$$

wherein it is assumed that, for some open subset $O \subseteq R^n$,

$$g : O \to R^n, \; H : O \to R^{nm}$$

are defined and continuously differentiable throughout O. Here R^{nm} denotes the space of $n \times m$ matrices with real entries. The differentiability of f with respect to both x and u for $x \in O$, $u \in R^m$, is then clear.

Our basic assumption is that the "uncontrolled" system

$$\dot{x} = f(x,0) = g(x) \tag{2.2}$$

has a solution $x(t)$ which is periodic with least positive period T_0. Beyond that we are concerned with the variational system based on $x(t)$, which is

$$\dot{\xi} = G(x(t))\xi + H(x(t))u \quad , \tag{2.3}$$

$$G(x(t)) \equiv \frac{\partial g}{\partial x}(x(t)) \quad \text{(Jacobian of } g \text{ w.r. to } x\text{).}$$

We denote by $\Phi(t,s)$ the fundamental matrix solution of

$$\frac{\partial \Phi(t,s)}{\partial t} = G(x(t))\Phi(t,s) \quad , \tag{2.4}$$

$$\Phi(s,s) = I \text{ ($n \times n$ identity matrix)} \tag{2.5}$$

and abbreviate $\Phi(t,0)$ by $\Phi(t)$. Both (2.3) and (2.4) may be taken to be defined for all real t if we extend $x(t)$ in the obvious way by periodicity. The resulting systems are periodically dependent on t with period T_0. The period transition map associated with the periodic solution $x(t)$ is the map defined by the matrix $\Phi(T_0)$.

Since the system (2.2) is autonomous, we may stipulate any point on the solution $x(t)$ as the initial point $x_0 = x(0)$. The assumption that $x(t)$ is a non-trivial periodic solution enables one to see readily that

$$p_1 = \dot{x}(0) = g(x_0) \neq 0 \quad . \tag{2.6}$$

It is well known that p_1 is an eigenvector of $\Phi(T_0)$ corresponding to the eigenvalue $\lambda_1 = 1$ of $\Phi(T_0)$. This is an immediate consequence of the fact that

$$\xi(t) \equiv \dot{x}(t) \equiv \Phi(t)p_1$$

is a periodic solution of $\dot{\xi} = G(x(t))\xi$. We assume that the remaining $n - 1$ eigenvalues of $\Phi(T_0)$, which we denote by $\lambda_2,\ldots,\lambda_n$, are all different from 1. An important special case arises when

$$|\lambda_i| < 1, \ i = 2,3,\ldots,n \quad , \tag{2.7}$$

in which case the periodic solution $x(t)$ of (2.2) is locally asymptotically stable. The condition (2.7) is not needed in this section but becomes very important in numerical considerations to be introduced later.

We denote by $\{p_1\}$ the one dimensional subspace of R^n spanned by p_1 and by P the $(n-1)$-dimensional subspace spanned by the (possibly generalized) eigenvectors $p_2, p_3,\ldots,p_n$ of $\Phi(T_0)$ associated with the eigenvalues $\lambda_2, \lambda_3,\ldots,\lambda_n$. If P is a real $n \times (n-1)$ matrix of rank $n - 1$ whose columns are $p_2, p_3,\ldots,p_n$, or consist of real and imaginary parts of $p_2, p_3,\ldots,p_n$ in the case of complex $\lambda_2, \lambda_3,\ldots,\lambda_n$, then each $p \in P$ has the unique representation

$$p = P\alpha, \ \alpha \in R^{n-1} \quad .$$

Let q_1 be the unique vector in R^n such that

$$q_1^* p_1 \; (\equiv (p_1, q_1)_{R^n}) = 1 \quad ,$$

$$q_1^* p = 0, \quad p \in P \quad .$$

Then q_1 is an eigenvector of $\Phi(T_0)^*$ corresponding to the eigenvalue 1. We denote by $\{q_1\}$ the one dimensional subspace of R^n spanned by q_1 and by Q the $(n-1)$-dimensional subspace of R^{n-1} spanned by the eigenvectors $q_2, q_3, \ldots, q_n$ associated with the eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ of $\Phi(T_0)^*$ (if we were to allow $\Phi(T)$ to be complex, $\lambda_2, \lambda_3, \ldots, \lambda_n$ would be replaced here by $\bar{\lambda}_2, \bar{\lambda}_3, \ldots, \bar{\lambda}_n$). Letting Q be the $n \times (n-1)$ matrix whose columns are $q_2, q_3, \ldots, q_n$ (or real and imaginary parts) it may be arranged that

$$Q^* P = I_{n-1} \quad .$$

Our requirement

$$\lambda_i \neq 1, \; i = 2, 3, \ldots, n \quad ,$$

guarantees that with

$$\theta(t) = Q^* (\Phi(t) - I_n) P \quad , \tag{2.8}$$

$\theta(T_0)$ is a nonsingular $(n-1) \times (n-1)$ matrix. Evidently we have the decompositions

$$\{p_1\} \oplus P = R^n = \{q_1\} + Q \quad .$$

Having taken care of the various definitions and assumptions needed, we turn now to consideration of periodic solutions of the controlled system (2.1) corresponding to controls

$$u \in L_m^2[0, T_1], \; T_1 > T_0 \quad ,$$

or, more precisely, to restrictions of such controls as explained below. The theorem which we present below may be proved much more simply when we require that $u \in C[0, T_1]$ by invoking the implicit function theorem and, additionally, a local uniqueness result is obtained in that case. Because of the method of proof employed, and in the interests of more colorful mathematical metaphor, we shall call this the "arrow and target" theorem. It will turn out that the proof rests on the intermediate value theorem.

Theorem 2.1. For arbitrarily small positive τ, δ, τ satisfying $\tau < T_1 - T_0$, there exists $\varepsilon = \varepsilon(\tau, \delta) > 0$ such that whenever

$$\|u\|_{L^2_m[0,T_1]} < \varepsilon \tag{2.9}$$

the system (2.1) has at least one solution $w(t)$ satisfying

$$\|w(t) - x(t)\| < \delta, \; 0 < t < T_1 \quad , \tag{2.10}$$

and such that

$$w(T) = w(0) \tag{2.11}$$

for at least one value of T with

$$|T - T_0| < \tau \quad . \tag{2.12}$$

Once w, T corresponding to u have been identified, w, u may be regarded as a periodic state, control pair by first restricting w and u to $[0,T]$ and then defining

$$w(t + kT) = w(t), \; u(t + kT) = u(t), \; k = \pm 1, \pm 2, \cdots \quad . \tag{2.13}$$

Proof of Theorem 2.1. For simplicity we begin with the equation

$$\dot{w}(t) = g(w(t)) + H(r(t))u \tag{2.14}$$

wherein $r(t)$ is a given continuous n-vector valued function defined on $[0,T]$ and satisfying

$$\|r(t) - x(t)\|_{R^n} < \delta, \; t \in [0,T_1] \quad ,$$

$x(t)$ being the periodic solution of (2.1) corresponding to $u(t) \equiv 0$ which we have discussed above. This includes, of course, the special case wherein

$$H(x(t)) \equiv H, \; H \text{ constant } n \times m \text{ matrix} \quad .$$

Once we have proved the existence of a periodic pair $w, u,$ for this special case we will be able to describe rather easily the modifications required to prove the corresponding result for (2.1).

Let $u \in L_m^2[0,T_1]$ satisfy (2.9) with ε yet to be determined, and let $w(t)$ be the solution of (2.1) corresponding to the control u and the initial state

$$w(0) = x_0 + p, \ p \in P, \ \|p\|_{R^n} < \rho \quad , \tag{2.15}$$

for some $\rho > 0$ yet to be determined. It is an easy consequence of the Caratheodory existence and regularity theory (see [G], e.g.) that we can assume

$$\|w(t) - x(t)\|_{R^n} < \hat{\delta} \tag{2.16}$$

for any $\hat{\delta} > 0$ by choosing ε and ρ in (2.9), (2.15) to be sufficiently small. Our intent is to keep u fixed and show that we can find p as in (2.15) and T satisfying (2.12) such that (2.11) holds.

Let $y(t)$ be the solution of

$$\dot{y}(t) = g(y(t)) + H(r(t))u(t) \quad , \tag{2.17}$$

$$y(0) = x_0 \quad . \tag{2.18}$$

Clearly $y(t)$ satisfies (2.16), i.e.,

$$\|y(t) - x(t)\|_{R^n} < \hat{\delta} \tag{2.19}$$

if ε is sufficiently small. Then

$$w(t) = y(t) + z(t) \tag{2.20}$$

where

$$\dot{z}(t) = g(y(t) + z(t)) - g(y(t)) \quad , \tag{2.21}$$

$$z(0) = p \quad . \tag{2.22}$$

The variational equation associated with the solution $y(t)$ is

$$\dot{\zeta}(t) = G(y(t))\zeta(t) \tag{2.23}$$

where $G(y(t)) = \frac{\partial g}{\partial x}(y(t))$ is the Jacobian of g evaluated along the solution y. If we let $\hat{\Phi}(t)$ be the matrix solution of

$$\dot{\hat{\Phi}}(t) = G(y(t))\hat{\Phi}(t), \quad t \in [0,T_1] \quad , \tag{2.24}$$

$$\hat{\Phi}(0) = I \tag{2.25}$$

and set

$$\hat{\theta}(t) = Q^*(\hat{\Phi}(t) - I)P \quad , \tag{2.26}$$

the usual regularity arguments show that $\hat{\theta}(t)$ *converges* uniformly to $\theta(t)$, given by(2.8), as $\|u\|_{L_m^2[0,T_1]}$ and hence $\sup_{t\in[0,T_1]} \{\|y(t) - x(t)\|_{R^n}\}$ tends to zero. Since $\theta(T_0)$ has been seen to be nonsingular, it follows that $\hat{\theta}(T)$ is invertible with $\hat{\theta}(T)^{-1}$ satisfying some bound

$$\|\hat{\theta}(T)^{-1}\| \le B \quad , \tag{2.27}$$

provided (2.9) and (2.12) hold with ε and τ sufficiently small.

The "target" of the "arrow and target" theorem is the image, Z_T, of the subspace $P \subseteq R^n$ under the map

$$Z(p,T) = z(p,T) - p \quad ,$$

where $z(p,t) = z(t)$ is the solution of (2.21) corresponding to the initial state (2.22), i.e., $z(p,0) = z(0) = p$. We can obtain a parametrized representation of Z_T by writing

$$z(p,T) - p = \alpha_1 p_1 + \hat{p}, \ \hat{p} \in P \quad , \tag{2.28}$$

$p_1 = \dot{x}(0) = g(x(0))$ as noted earlier and (since $\hat{p} \in P$) setting

$$\hat{p} = P\hat{\alpha}, \ \hat{\alpha} = \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in R^{n-1} \quad , \tag{2.29}$$

with P the $n \times (n-1)$ matrix described earlier. Then

$$\alpha_1 = q_1^*(z(p,T) - p) \quad , \tag{2.30}$$

$$\alpha = Q^*(z(p,T) - p) \tag{2.31}$$

and writing

$$p = P\beta, \ \beta \in R^{n-1} \quad ,$$

we see that the Jacobian of $\hat{\alpha}$ with respect to β at $\beta = 0$ (i.e., $p = 0$) is precisely $\hat{\theta}(T)$ as given by (2.26). From this and the implicit function theorem it follows that Z_T can be represented, for p near 0 and T near T_0, as corresponding to the set of points (2.30), (2.31) for which

$$\alpha_1 = \varphi(\hat{\alpha}, T) \quad ,$$

with φ of class C^1 near $\hat{\alpha} = 0$, $T = T_0$. In fact (2.30) shows that

$$\frac{\partial \varphi}{\partial \hat{\alpha}}(0,T_0) = q_1^* \frac{\partial}{\partial p}(z(p,T_0) - p)\Big|_{p=0} \frac{\partial p}{\partial \beta} \frac{\partial \beta}{\partial \hat{\alpha}}\Big|_{\hat{\alpha}=0}$$

$$= q_1^*(\hat{\Phi}(T_0) - I)P\hat{\theta}(T_0)^{-1} \quad . \tag{2.32}$$

The surface Z_T will thus be nearly tangent to the subspace P at $p = 0$ for small $\|u\|$, i.e., for $y(t)$ near $x(t)$, and T near T_0, because $\Phi(T_0)P = P\Lambda$ for an appropriate $(n-1) \times (n-1)\Lambda$ and $q_1^*P = 0$.

The "arrow" of our theorem is the vector $x_0 - y(T)$. The "arrow" hits the "target" just in case there is a vector $p \in P$ and an instant T such that

$$x_0 - y(T) = y(0) - y(T) = z(p,T) - p$$
$$= z(p,T) - z(p,0) = z(T) - z(0) \quad ,$$

for then

$$w(0) = y(0) + z(0) = y(T) + z(T) = w(T) \quad . \tag{2.33}$$

That such a "hit" takes place is a consequence of the intermediate value theorem for continuous functions on R^1. The situation in hand is represented graphically in Figure 2.1 where + • + • + is the trajectory of $x_0 - x(T)$

for T near T_0, $\longrightarrow\!\!\!\!\longrightarrow\!\!\!\!\longrightarrow$ is the trajectory of $x_0 - y(T)$, the hyperplane P is shown, and the curves - - - - - indicate the outline of a cylindrical region

$$\|x - (x_0 - x(T))\| < d \quad , \tag{2.34}$$

for $d < \delta$, truncated by hyperplanes parallel to P

$$H^{\pm} = \{x \mid x = \gamma p_1 + p,\ p \in P,\ p_1^* x (= \gamma \|p_1\|^2) = \pm L\} \quad .$$

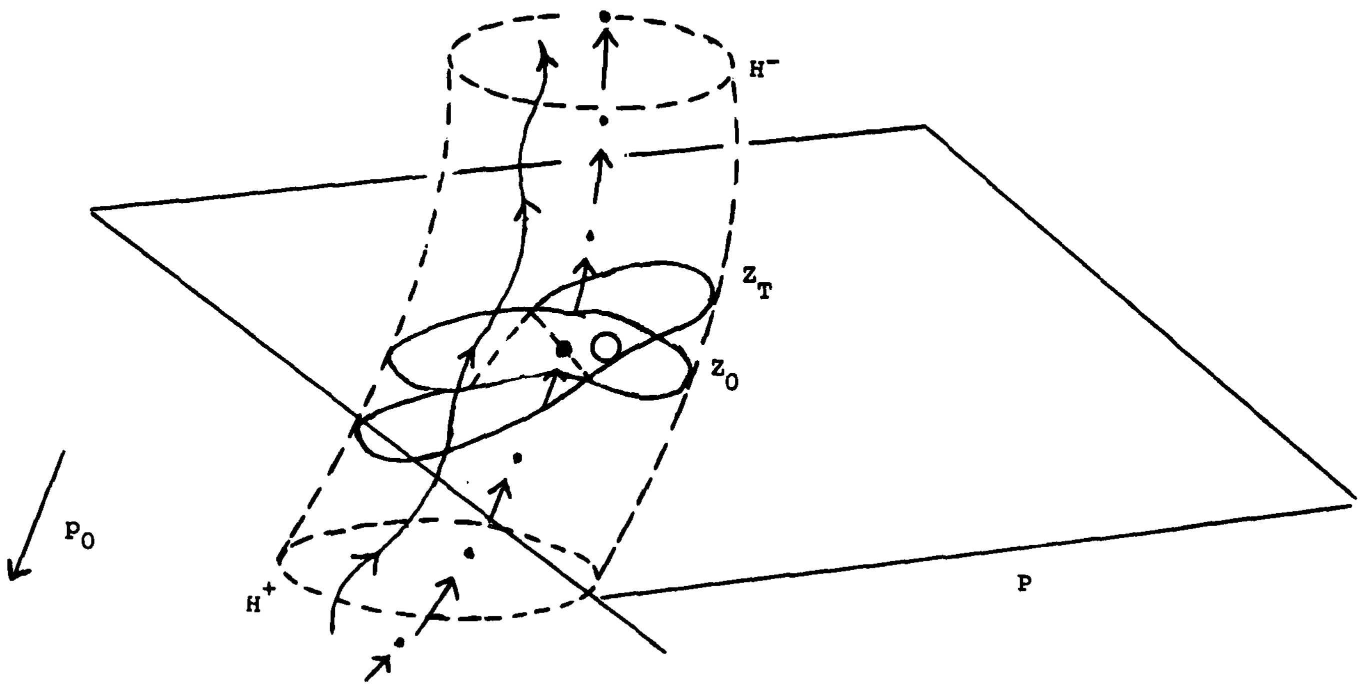

Figure 2.1

The surface Z_T depends on u via the dependence of z on y in equation (2.21). Indicating this dependence by $Z_T(u)$, we begin by taking d small enough in (2.34) so that $Z_{T_0}(0)$ bisects the cylinder into two regions. This must be the case for d small because the tangent vector to

$x_0 - x(T)$ at $T = T_0$ is p_1, while the tangent hyperplane to $Z_{T_0}(0)$ at the origin is P. Next, L and τ are restricted so that

$$p_1^*(x_0 - x(T)) > 2L,\ T_0 - 2\tau \leq T < T_0 - \tau \quad ,$$
$$p_1^*(x_0 - x(T)) < -2L,\ T_0 + \tau < T \leq T_0 + 2\tau \quad .$$

Then we make ε, and hence $\|u\|_{L_m^2[0,T_1]}$, small enough so that $x_0 - y(T)$ lies in the cylinder (2.34) for $|T - T_0| \leq \tau$ and

$$p_1^*(x_0 - y(T)) > L,\ T_0 - 2\tau \leq T < T_0 - \tau \quad , \tag{2.35}$$
$$p_1^*(x_0 - y(T)) < -L,\ T_0 + \tau < T \leq T_0 + 2\tau \quad , \tag{2.36}$$

both of which are possible because $y(t)$ converges uniformly to $x(t)$ as $\|u\|_{L_m^2[0,T_1]}$ tends to zero. Finally, ε is further restricted, if necessary,

so that $Z_T(u)$ bisects the cylinder (2.34) when $\|u\|_{L_m^2[0,T_1]} \leq \varepsilon$. Because $z(0,T) \equiv 0$, $Z(0,T) \equiv 0$ and the surface $Z_T(u)$ always passes through the origin. Then $\frac{\partial \varphi}{\partial \alpha}(p,T)$ can be uniformly bounded for p small and T near T_0, using (2.27), (2.32), and, further restricting d if necessary, the intersection of $Z_T(u)$ with the cylinder (2.34) does not meet the ends, $H^{\pm}$, of the cylinder for $\|u\|_{L_m^2[0,T_1]} \leq \varepsilon$, $|T - T_0| \leq \tau$. Put another way, if $P_1(T)$ denotes the connected neighborhood of the origin in P which maps into the cylinder (2.34) under $Z(\cdot,T)$,

$$|p_1^* Z(p,T)| < L \text{ for } \|u\|_{L_m^2[0,T_1]} \leq \varepsilon,\ |T - T_0| \leq \tau,\ p \in P_1(T) \quad . \tag{2.37}$$

It follows that we may unambiguously denote the two *components of the cylinder* (2.34) cut out by Z_T by $C^+(T)$, $C^-(T)$, according to whether that component contains H^+ or H^-, respectively. For x in the cylinder (2.34) and $T_0 - 2\tau \leq T \leq T_0 + 2\tau$ we define

$$D(x,T) = \pm\ \mathrm{dis}(x, Z_T) \quad ,$$

the distance being the distance within the cylinder (2.34) and $+$ being used in $C^+(T)$, $-$ in $C^-(T)$. Then consider the function $D(x_0 - y(T),T)$. From the continuity of $D(x,T)$ (the proof of which we omit here, but it is not really difficult to establish) it follows that $D(x_0 - y(T),T)$ is a continuous function of T. From (2.35), (2.36) it follows that $D(x_0 - y(T),T)$ is positive for $T_0 - 2\tau < T < T_0 - \tau$, negative for $T_0 + \tau < T < T_0 + 2\tau$. From the intermediate value theorem there exists at least one T in $T_0 - \tau < T < T_0 + \tau$ such that $D_0(x_0 - y(T),T) = 0$, from which it follows that

$$x_0 - y(T) \in Z_T$$

for such T and, as indicated earlier, this shows that there exists $w(t)$ a solution of (2.14) such that

$$w(0) = w(T)$$

and we have the result for the system (2.14).

We indicate only briefly how the result is extended to the system (2.1), i.e.,

$$\dot{w} = g(w) + H(w)u \quad . \tag{2.1r}$$

The main complication is that when we let

$$\dot{y} = g(y) + H(y)u, \; y(0) = x_0 \quad , \tag{2.38}$$

then, in order that $w = y + z$ should be a solution of (2.1), we must have

$$\dot{z} = g(y+z) - g(y) + (H(y+z) - H(y))u \tag{2.39}$$

so that small solutions $z(t)$ of (2.21) do not approximately satisfy (2.23)· but rather the equation

$$\dot{\zeta} = [G(y(t)) + G_u(t,y(t))]\,\zeta \tag{2.40}$$

where

$$G_u(t,x) = \frac{\partial(H(x)u(t))}{\partial x}$$

is an $n \times n$ matrix function of t, x which, for each x, is a linear function of $u(t)$. It is not hard to show that

$$\|G_u(t,x)\| \leq \gamma\|u(t)\|, \quad t \in [0,T], \; x \in C \quad , \qquad (2.41)$$

where C is a compact subset of O including a neighborhood of the periodic solution $x(t)$ and γ is a positive constant. If we denote the fundamental solution matrix for (2.40) which reduces to the identity at $t = 0$ by $\Phi_u(t)$, then

$$\dot{\Phi}_u(t) = G(y(t))\Phi_u(t) + G_u(t,y(t))\Phi_u(t)$$

and the variation of parameters formula applies to give, with $\hat{\Phi}(t)$ as in (2.24),

$$\begin{aligned}\Phi_u(t) &= \hat{\Phi}(t)\Phi_u(0) + \int_0^t \hat{\Phi}(t,s)G_u(s,y(s))\Phi_u(s)ds \\ &= \hat{\Phi}(t) + \int_0^t \hat{\Phi}(t,s)G_u(s,y(s))\Phi_u(s)ds\end{aligned}$$

from which, using (2.40), a very easy argument shows that

$$\lim_{\|u\|_{L^2_m[0,T_1]} \to 0} \Phi_u(t) = \hat{\Phi}(t) \quad .$$

Then all of the arguments set forth earlier for (2.14) apply with $\hat{\Phi}(t)$ replaced by $\Phi_u(t)$, provided $\|u\|_{L^2_m[0,T_1]}$ is sufficiently small, to establish again the existence of p and T so that the solution $w(t)$ of (2.1) with $w(0) = x_0 + p$ satisfies (2.11). With this we may regard the proof of Theorem 2.1 to be complete.

The next step is to establish, rather easily, a certain localization result for the periodic solutions $w(t)$ resulting from control functions $u(t)$ with $\|u\|_{L^2_m[0,T_1]}$ small. Let us note, as in (2.19) earlier, that, given any $\hat{\delta} > 0$, we can assume

$$\|y(t) - x(t)\| < \hat{\delta},\ t \in [0,T_1] \quad ,$$

where $y(t)$ is now the solution of (2.38), by taking $\|u\|_{L^2_m[0,T_1]} < \varepsilon$, ε sufficiently small. Then, since $x(0) = x(T_0)$, we can further assume, given $\Delta > 0$, that

$$\|y(0) - y(T)\| < \Delta,\ T_0 - \tau < T < T_0 + \tau \quad , \tag{2.42}$$

provided both ε and τ are taken sufficiently small.

Now consider the map $Z(p,T) = z(p,T) - p$ with $z(p,t) = z(t)$ the solution of (2.39) satisfying

$$z(0) = p \in P \quad .$$

Since $\partial(z(p,T) - p)/\partial p$ is the restriction of the map $\Phi_u(T) - I$ to P, and since $p_1 \in P$ is (modulo scalar multiplications) the only null vector of $\Phi(T) - I$, it follows that there are positive numbers ρ and μ such that for $\|u\|_{L^2_m[0,T_1]} < \varepsilon$,

$$\|z(T) - z(0)\| = \|z(p,T) - p\| > \mu\|p\|$$

for $\|p\| < 2\rho$, $T_0 - \tau < T < T_0 + \tau$. Then, by taking

$$\Delta < \mu\rho$$

in (2.42) (which may involve further restriction of ε and τ) we see that we cannot have

$$y(T) + z(T) = w(T) = w(0) = y(0) + z(0) \tag{2.43}$$

for $z(0) = x_0 + p$, $p \in P$, $\rho < \|p\| < 2\rho$, since (2.43) would give

$$y(T) - y(0) = z(0) - z(T)$$

which is impossible since

$$\|y(T) - y(0)\| < \Delta < \mu\rho < \mu\|p\| < \|z(T) - z(0)\|$$

for such p.

Equally well, the proof of Theorem 2.1 shows that for small $\|p\|$, $p \in P$, and appropriately selected $\tau > 0$, no solution $w(t)$ originating at $w(0) = x_0 + p$ and corresponding to a control u with $\|u\|_{L^2_m[0,T_1]} < \varepsilon$, ε sufficiently small, can have a period T with T in the ranges $T_0 - 2\tau < T < T_0 - \tau$, $T_0 + \tau < T < T_0 + 2\tau$. We can put all of this together in the following "localization" result.

Proposition 2.2. *If* ε *is sufficiently small we can find positive* τ *and* ρ *such that when* $\|u\|_{L^2_m[0,T_1]} < \varepsilon$

(i) *equation* (2.1) *has a solution* $w(t)$ *periodic with period* T *and with* $w(0) = x(0) + p$,

$$(p,T) \in N(\rho,\tau) = \{p \in P, \|p\|, T_0 - \tau < T < T_0 + \tau\}$$

(ii) *equation* (2.1) *has no solutions of period* T *and initial state* $w(0) = x_0 + p$ *if*

$$(p,T) \in N(2\rho,2\tau) - N(\rho,\tau) \quad .$$

The localization result is quite important in the study of the optimal control problem introduced below because it enables us to single out a set of triples x, u, T which may be confined to a bounded region simply by taking $\|u\|_{L^2[0,T]}$ sufficiently small, rather than by introducing further, extraneous, constraints into the problem.

It seems fairly clear that Theorem 2.1 and Proposition 2.2 can be extended to the general system (1.11) provided that $\frac{\partial f}{\partial u}(x,u)$ remains uniformly bounded for $x \in C \subseteq O$, $u \in E^m$, where C is a compact subset of E^n containing the trajectory $x(t)$, $0 \le t \le T_0$, in its interior.

We proceed, now, to introduce and study the optimal control problem referred to in Section 1. For the moment we replace the general cost integrand $w(x,u)$ of (1.12) by a less general expression, augmented by a parameter σ,

$$W(\sigma,x) + u^* Uu \quad .$$

It should not be difficult to extend the work to more general integrands $w(\sigma,x,u)$.

Let the system (2.1), i.e.,

$$\dot{x} = g(x) + H(x)u, \; x \in E^n, \; u \in E^m$$

have the properties set forth preparatory to and in Theorem 2.1 and let $x(t)$ be a periodic solution with least positive period T_0 of

$$\dot{x} = g(x)$$

having the properties developed earlier. Let U be a positive definite $m \times m$ matrix and let $W(\sigma,x)$ be a continuous function of σ,x for $\sigma \geq 0$ and $x \in O \subseteq R^n$ such that

$$W(0,x(t)) \equiv 0, \; t \in [0,T_0] \quad , \qquad (2.44)$$

$$W(\sigma,x) \geq 0, \; \sigma \geq 0, \; x \in O \quad , \qquad (2.45)$$

$$W(\sigma,x(t)) \not\equiv 0, \; \sigma > 0, \; t \in [0,T_0] \quad . \qquad (2.46)$$

For each trajectory, control and period triple w, u, T with $w(t)$, $u(t)$ defined on the interval $[0,T_1]$, and $w(0) = w(T)$, $0 < T \leq T_1$, we define the cost functional

$$J(\sigma,w,u,T) = \frac{1}{T} \int_0^T [W(\sigma,w(t)) + u(t)^* Uu(t)]dt \quad . \qquad (2.47)$$

We require $T_1 > T_0$ and we denote by M the set of trajectory, control and period triples w, u, T, defined for $0 \leq t \leq T_1$, $u \in L_m^2[0,T_1]$, such that w, u satisfy (2.1) on $[0,T_1]$, $0 < T \leq T_1$, and

$$w(0) = w(T) \quad .$$

There may be several, or even infinitely many values of T corresponding to a given pair w, u. Theorem 2.1 shows that, given the existence of the periodic solution $x(t)$, with period T_0, corresponding to the control $u(t) \equiv 0$ with appropriate assumptions on $\Phi(T_0)$, M includes at least one triple w, u, T for every $u \in L_m^2[0,T_1]$ with $\|u\|_{L_m^2[0,T_1]}$ sufficiently small. We make M into a metric space by defining the distance function

$$d(w,u,T;\ \breve{w},\breve{u},\breve{T}) = d_H(R(w),R(\breve{w})) + \|u-\breve{u}\|_{L^2[0,T_1]} + |T - \breve{T}|$$

where $R(w) = \{w = E^n \mid w = w(t)$ for some $t \in [0,T_1]\}$ and d_H is the usual Haussdorff metric on compact sets.

Property (2.46) of the function W shows that if $\sigma > 0$

$$J(\sigma,x,0,T_0) \equiv J_0(\sigma) > 0 \quad ,$$

and the continuity of W with respect to σ shows that

$$\lim_{\sigma\to 0} J_0(\sigma) = 0 \quad .$$

For $\sigma > 0$ we define M_σ to be the subset of M for which

$$J(\sigma,w,u,T) < J_0(\sigma) \quad .$$

If w, u $T \in M_\sigma$, it is clear that

$$\frac{1}{T}\int_0^T u(t)^* U u(t)dt < J_0(\sigma) \quad , \tag{2.48}$$

which implies that

$$\int_0^T \|u(t)\|^2 dt < \frac{T\,J_0(\sigma)}{\mu_1} < \frac{T_1 J_0(\sigma)}{\mu_1} \tag{2.49}$$

when μ_1 is the smallest eigenvalue of the symmetric positive definite matrix U.

The localization result expressed in Proposition 2.2 shows that the set of all w, u, T in M for which $\|u\|_{L^2_m[0,T_1]} < \varepsilon$ may, if ε is sufficiently small, be decomposed in a natural way into two components: those lying in $N(\rho,\tau)$ and those lying outside $N(2\rho,2\tau)$, there being a triple w, u, T in $N(\rho,\tau)$ for every u in $L^2_m[0,T_1]$ with $\|u\|_{L^2_m[0,T_1]} < \varepsilon$. The importance of this result lies in the fact that it provides us with _a priori_ bounds on a certain subset of the periodic orbits associated with $u \in L^2_m[0,T_1]$ of small norm without the necessity of imposing any additional constraints.

Theorem 2.3. _Let_ $\varepsilon > 0$ _be sufficiently small and let_ ρ, τ _be such that Proposition_ 2.2 _applied for_ $\|u\|_{L^2_m[0,T_1]} < \varepsilon$. _Let_ $\sigma > 0$ _be small enough so that_ (cf. (2.49))

$$J_0(\sigma) < \frac{\mu_1 \varepsilon}{T_1} . \tag{2.50}$$

Then there exists at least one element $\hat{w}, \hat{u}, \hat{T}$ _in_ $N(\rho,\tau)$ _such that_

$$J(\sigma,\hat{w},\hat{u},\hat{T}) \le J(\sigma,w,u,T), \quad w,u, T \in N(\rho,\tau) .$$

Proof. The Caratheodory regularity theory referred to earlier allows us to see that if $\varepsilon > 0$ is sufficiently small and w, u, T is in $N(\rho,\tau)$ with $\|u\|_{L^2_m[0,T_1]} < \varepsilon$, then there exists $B(\varepsilon) > 0$ such that

$$\|w(t) - x(t)\| \le B(\varepsilon), \quad t \in [0,T_1] .$$

Moreover,

$$\lim_{\varepsilon \to 0} B(\varepsilon) = 0 .$$

In particular, then, there is some $M > 0$ and a closed subset $C \subseteq O \cap \{w \mid \|w\| \le M\}$ such that

$$w(t) \in C, \quad t \in [0,T_1] ,$$

for such w, u, T. Using the Schwarz inequality we quickly see that

$$\|w(t) - w(t+\delta t)\| = \|\int_t^{t+\delta t} (g(w(s)) + H(w(s))u(s))ds\|$$

$$< g_1\delta t + h_1 \int_t^{t+\delta t} \|u(s)\|ds$$

$$< g_1\delta t + h_1 \left(\int_t^{t+\delta t} \|u(s)\|^2 ds\right)^{1/2} \left(\int_t^{t+\delta t} 1\ ds\right)^{1/2}$$

$$< g_1\delta t + h_1\varepsilon\ \delta t^{1/2}$$

where

$$g_1 = \sup_{w\in C} \|g_1(w)\| \quad ,$$

$$h_1 = \sup_{w\in C} \|H(w)\| \quad .$$

Since (2.50) is true, (2.49) shows that we may assume a minimizing sequence w_k, u_k, T_k such that (starting with $k = 2$ to avoid confusion with T_1 already defined)

$$w_k, u_k, T_k \in N(\rho,\tau),\ k = 2,3,\cdots,$$

$$\|u_k\|_{L_m^2[0,T_1]} < \frac{T_1 J_0(\sigma)}{\mu_1} < \varepsilon,\ k = 2,3,\cdots,$$

$$\lim_{k\to\infty} J(\sigma,w_k,u_k,T_k) = \inf_{w,u,T\in N(\rho,\tau)} J(\sigma,w,u,T) \quad . \qquad (2.51)$$

From the boundedness and equicontinuity of the w_k, the boundedness of the T_k and the weak compactness of $\{u \in L_m^2[0,T_1] \mid \|u\|_{L_m^2[0,T_1]} < \frac{T_1 J_0(\sigma)}{\mu_1}\}$ we conclude that we can find w, u, T in $N(\rho,\tau)$ and a subsequence of $\{w_k,u_k,T_k\}$ which, for convenience, we will still call $\{w_k,u_k,T_k\}$ such that

$$\lim_{k\to\infty}\left(\sup_{t\in T_1}\|w_k(t)-\hat{w}(t)\|\right)=0\quad, \tag{2.52}$$

$$\lim_{k\to\infty} T_k = \hat{T} \tag{2.53}$$

and $\{u_k\}$ converges weakly to $\hat{u}$ in $L_m^2[0,T]$. Combining (2.52) and (2.53) it is clear that

$$\lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k} W(\sigma,w_k(t))dt = \frac{1}{\hat{T}}\int_0^{\hat{T}} W(\sigma,w(t))dt \quad .$$

From the weak convergence of $\{u_k\}$ to $\hat{u}$ in $L_m^2[0,T_1]$ we immediately have the weak convergence of $\{u_k\}$ to $\hat{u}$ in $L_m^2[0,\hat{T}]$. Then, since for any $v \in L_m^2[0,T_1]$

$$\left|\int_{\hat{T}}^{T_k} v(t)^* u_k(t)dt\right| \leq \left(\int_{\hat{T}}^{T_k} \|u_k(t)\|^2 dt\right)^{1/2}$$

$$\times\left(\int_{\hat{T}}^{T_k}\|v(t)\|^2 dt\right)^{1/2} \leq \varepsilon\|v\|_{L_m^2[\hat{T},T_k]}$$

(or $L_m^2[T_k,\hat{T}]$ if $T_k < \hat{T}$) and, as is well known

$$\lim_{T_k\to\hat{T}}\int_{\hat{T}}^{T_k} f(t)dt = 0$$

for any integrable $f(t)$, we conclude that if the $u_k(t)$ are redefined to be zero in $[T_k,T_1]$, $k = 2,3,\cdots$ and $\hat{u}(t)$ is redefined to be zero in $[\hat{T},T_1]$, it will still be true that $\{u_k\}$ converges weakly to $\hat{u}$. Then, using a result in [H] (Theorem 8, p. 209) we know that

$$\int_0^{\hat{T}} u(t)^* Uu(t)dt \leq \lim_{k\to\infty}\int_0^{T_k} u_k(t)^* Uu_k(t)dt$$

and we see that

$$J(\sigma,\hat{w},\hat{u},\hat{T}) \leq \lim_{k\to\infty} J(\sigma,w_k,u_k,T_k)$$

and hence, from (2.51), that

$$J(\sigma,\hat{w},\hat{u},\hat{T}) = \inf_{w,u,T\in N(\rho,\tau)} J(\sigma,w,u,T)$$

$$= \lim_{k\to\infty} J(\sigma,w_k,u_k,T_k)$$

and the theorem is proved.

It will be seen in the course of the discussion in Section 3 that the above optimal control formulation is not adequate for all purposes. In some cases we need to take a fixed cost functional

$$J(x,u,T) = \frac{1}{T}\int_0^T [W(x(t)) + u(t)^* Uu(t)]dt$$

where, assuming the system (2.1) now augmented, as in Section 1, with a parameter θ, viz.:

$$\dot{x} = f(x,\theta,u) \tag{2.54}$$

or

$$\dot{x} = g(x,\theta) + H(x,\theta)u \quad , \tag{2.55}$$

we suppose that $W(\tilde{x}(t)) \equiv 0$ on a $\tilde{T}$-periodic solution $\tilde{x}(t)$ of the "uncontrolled" system (2.54) or (2.55) with $u \equiv 0$ and $\theta = \tilde{\theta}$, where $\tilde{\theta}$ is some minimal value of the parameter μ, and that

$$W(x) > 0, \; x \notin \{\tilde{x}(t) \mid 0 \leq t \leq \tilde{T}\} \quad .$$

The optimal solution for $\theta = \tilde{\theta}$ is clearly $\tilde{x}(t)$, $0 \leq t \leq \tilde{T}$. One wishes to modify Theorem 2.3 to establish the existence, locally, of an optimal solution for θ in some neighborhood of $\tilde{\theta}$.

The only modifications that are at all significant here involve modification of the localization result, Proposition 2.2, so that it applies for $\|u\|_{L_m^2[0,T_1]}$ sufficiently small and θ sufficiently near $\tilde{\theta}$ and replacement of the *a priori* bound (2.48) on u by

$$\frac{1}{T}\int_0^T u(t)^* Uu(t)dt < \frac{1}{T_\mu}\int_0^{T_\mu} W(x_\mu(t))dt$$

where $x_\mu(t)$ is some periodic solution of

$$\dot{x}_\mu(t) = f(x_\mu(t),\mu,0)$$

lying near the periodic solution $\tilde{x}(t)$ with period T_μ near $\tilde{T}$ - the existence of which is established much as in Theorem 2.1, except that the proof can be greatly simplified now, using the implicit function theorem since, regarded as a control, μ is now (trivially) continuous in t. In fact, x_μ and T_μ are unique in this case. The rest of the argument is almost word for word the same as in Theorem 2.3 above and will not be repeated here.

3. Necessary Conditions for Optimality; Computational Considerations

In the present section we consider the system

$$\dot{x} = f(x,u),\ x \in O \subseteq E^n,\ u \in E^m \quad , \tag{3.1}$$

wherein it is assumed that $f(x,u)$ is continuously differentiable with respect to both x and u over the range indicated. We suppose that $x(t)$ is a periodic solution of (3.1), with least positive period $\hat{T}$, which corresponds to the control $\hat{u}(t)$ on $[0,\hat{T}]$. For the moment we make no further assumptions on f but restrict attention to piecewise continuous $\hat{u}$ and piecewise continuous variations δu from $\hat{u}$, but fairly minor modifications allow us, in the case of

$$\dot{x} = g(x) + H(x)u,\ x \in O \subseteq E^n,\ u \in E^m \quad , \tag{3.2}$$

the system treated in Section 2, to also admit the possibility that $\hat{u}$ and the variations δu are just square integrable.

A number of authors have considered necessary and sufficient conditions for periodic systems ([I], [J], [K], [L]). We do not develop the conditions here because they are new - they are not - but to allow us to make appropriate reference later in this section. We seek necessary conditions in order that the control u should afford a local minimum (at least) for the cost functional

$$J(x,uT) = v(x(0)) + \frac{1}{T}\int_0^T w(x(t),u(t))dt \quad , \tag{3.3}$$

where $v(x)$ and $w(x,u)$ are continuously differentiable in x, (x,u) respectively, for $x \in O \subseteq E^n$, $u \in E^m$. In (3.3) it is assumed that $x(t)$ is a periodic solution of (3.1) having period near the period $\hat{T}$ for $\hat{x}(t)$:

$$|T - \hat{T}| < \tau,\ \tau > 0 \quad ,$$

that $x(t)$ lies near $\hat{x}(t)$ in the sense that

$$\|x(t) - \hat{x}(t)\| < \varepsilon,\ 0 < t < \sup(T,\hat{T})$$

for some $\varepsilon > 0$ and that

$$\|u(t) - \hat{u}(t)\| < \delta, \ 0 < t < \sup(T,\hat{T}) \quad ,$$

for some $\delta > 0$. In the case of (3.2) this can be relaxed to

$$\|u - \hat{u}\|_{L^2[0,T_1]} < \delta, \ T_1 > \max\{T,\hat{T}\} \quad .$$

Let $\delta u(t)$ be piecewise continuous for $0 < t < T_1$, $T_1 > \hat{T}$. We consider controls

$$u(t) = \hat{u}(t) + \alpha\, \delta u(t), \ \alpha \text{ real} \quad , \tag{3.4}$$

where $\hat{u}(t)$ is extended outside $[0,\hat{T}]$ by periodicity as required.

We suppose the time parameter normalized in such a way that $\hat{u}(t)$ is continuous at $t = 0$ (equivalently at $\hat{T}$) and we set

$$F(t) = F(\hat{x}(t), \hat{u}(t)) = \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t)), \ 0 < t < \hat{T} \quad . \tag{3.5}$$

As earlier in the paper, we denote by $\Phi(t,s)$ the matrix solution of

$$\frac{d\Phi(t,s)}{ds} = F(t)\Phi(t,s), \ \Phi(s,s) = I \quad ,$$

and we abbreviate $\Phi(t,0)$ by $\Phi(t)$.

The variational theory for a <u>controlled</u> periodic solution $\hat{x}(t)$ is somewhat different from that for an uncontrolled periodic solution $x(t)$ as discussed earlier in this paper. If we let

$$p = f(\hat{x}(0), \hat{u}(0)) \tag{3.6}$$

it is no longer necessarily true that p is an eigenvector of $\Phi(\hat{T})$ corresponding to an eigenvalue $\lambda_1 = 1$. To see why this is so, consider, e.g., the case where $\hat{u}(t)$ is periodic and of class C^1 on $[0,\hat{T}]$. Let

$$p(t) = \dot{\hat{x}}(t) = f(\hat{x}(t), \hat{u}(t)), \ 0 < t < \hat{T} \quad .$$

Differentiating (3.1) gives

$$\dot{p} = F(t)p + H(t)\dot{\hat{u}}(t) \quad , \tag{3.7}$$

where

$$H(t) = H(\hat{x}(t), \hat{u}(t)) = \frac{\partial f}{\partial u}(\hat{x}(t), \hat{u}(t)) \quad , \tag{3.8}$$

so that

$$p(t) = \Phi(t)p(0) + \int_0^t \Phi(t,s)H(s)\dot{\hat{u}}(s)ds \quad .$$

Since $p(\hat{T}) = \dot{\hat{x}}(\hat{T}) = \dot{\hat{x}}(0) = p(0) = p$, we have

$$(I - \Phi(\hat{T}))p = \int_0^{\hat{T}} \Phi(\hat{T},s)H(s)\dot{\hat{u}}(s)ds \tag{3.9}$$

and p is an eigenvector of $\Phi(\hat{T})$ corresponding to the eigenvalue 1 if and only if the right hand side of (3.9) turns out to be zero. Of course, in the uncontrolled case the right hand side is automatically equal to zero.

We are going to proceed under the following

Assumption: Either (i): p, given by (3.6), is the unique eigenvector of $\Phi(\hat{T})$ corresponding to the eigenvalue 1, which is simple, or (ii): $\Phi(\hat{T})$ has no eigenvalue equal to 1.

It is not particularly difficult to treat the case where $\Phi(\hat{T})$ may have a simple eigenvalue 1 with corresponding eigenvector different from p but we shall not do so here.

In case (i) the study of perturbed periodic solutions corresponding to a small perturbation $\delta u(t)$ from the control $\hat{u}(t)$ is just Theorem 2.1. In case (ii) one can establish the following

Proposition 3.1: If $\Phi(\hat{T})$ has no eigenvalue equal to 1 then, corresponding to each piecewise continuous perturbed control

$$u(t) = \hat{u}(t) + \alpha\, \delta u(t)$$

with $|\alpha|$ sufficiently small, there is a unique perturbation $\delta x(0)$ of the initial state so that the solution $x(t)$ of (3.1) corresponding to the control $u(t)$ and the initial state $x(0) = \hat{x}(0) + \delta x(0)$ remains periodic with period $\hat{T}$.

We omit the proof. A comparable result can be obtained for (3.2) with δu in $L^2[0,\hat{T}]$.

To proceed, we consider a perturbed control (3.4), which we take to be piecewise continuous in this discussion, and we let

$$x(t,\alpha) = \hat{x}(t) + \alpha\,\delta x(t) + o(|\alpha|),\ |\alpha| \to 0 \qquad (3.10)$$

be a perturbed periodic solution near $\hat{x}(t)$ (uniqueness fails under case (ii) of our Assumption even with piecewise continuous controls) with period

$$T(\alpha) = \hat{T} + \alpha\,\delta T + o(|\alpha|),\ |\alpha| \to 0 \ .$$

In (3.10) $\delta x(t)$ is a solution of the variational equation (cf. (3.5), (3.7), (3.8))

$$\delta\dot{x}(t) = F(t)\delta x(t) + H(t)\delta u(t) \qquad (3.11)$$

for $0 \leq t \leq \hat{T}$, and the equation is extended outside this range, as required, using the T periodicity of $\hat{x}(t)$ and $\hat{u}(t)$.

Denoting the gradients of v, w with respect to x, (x,u), respectively, by $\frac{\partial v}{\partial x}$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial u}$, and treating these as row vectors, it may be seen that

$$\begin{aligned} J(x(\cdot,\alpha),\ u(\cdot,\alpha),\ T(\alpha)) &= J(\hat{x},\hat{u},\hat{T}) \\ &+ \Big\{\frac{\partial v}{\partial x}(\hat{x}(0))\delta x(0) + \frac{1}{\hat{T}}\int_0^{\hat{T}} \Big[\frac{\partial w}{\partial x}(\hat{x}(t),\ \hat{u}(t))\delta x(t) \\ &+ \frac{\partial w}{\partial u}(\hat{x}(t),\ \hat{u}(t))\delta u(t)\Big]dt + \Big(-\frac{1}{\hat{T}^2}\int_0^{\hat{T}} w(\hat{x}(t),\ \hat{u}(t))dt \\ &+ \frac{1}{T} w(\hat{x}(0),\ \hat{u}(0))\Big)\delta T\Big\}\ \alpha + o(|\alpha|),\ \alpha \to 0 \end{aligned} \qquad (3.12)$$

wherein we have used periodicity to see that

$$w(\hat{x}(\hat{T}),\ \hat{u}(\hat{T})) = w(\hat{x}(0),\ \hat{u}(0)) \ .$$

Now the variational derivatives δu, δx and δT are not independent; the last two are determined by δu and the normalization of $\delta x(0)$. In order to be able to express the cost variational derivative explicitly in terms of

δu, rather than implicitly through δx and δT, one introduces a solution $\lambda(t)$ of the "adjoint system"

$$\dot{\lambda}(t) = -F(t)^{*}\lambda(t) + \frac{1}{\hat{T}}\frac{\partial w}{\partial x}(\hat{x}(t),\ \hat{u}(t)) \quad . \tag{3.13}$$

The boundary conditions will be specified shortly. We find then that

$$\lambda(\hat{T})^{*}\delta x(T) - \lambda(0)^{*}\delta x(0) = \int_0^{\hat{T}} \frac{d}{dt}(\lambda(t)^{*}\delta x(t))dt$$

$$= (\text{cf. } (3.11),\ (3.13)) = \int_0^{\hat{T}} [(-\lambda(t)^{*}F(t) + \frac{1}{T}\frac{\partial w}{\partial x}(\hat{x}(t),\ \hat{u}(t))\delta x(t)$$

$$+ \lambda(t)^{*}(F(t)\delta x(t) + H(t)\delta u(t))]dt$$

$$= \int_0^{\hat{T}} \lambda(t)^{*}H(t)\delta u(t)dt + \frac{1}{\hat{T}}\int_0^{\hat{T}} \frac{\partial w}{\partial x}(\hat{x}(t),\ \hat{u}(t))\delta x(t)dt \quad . \tag{3.14}$$

To first order, the periodicity requirement

$$x(\hat{T} + \delta T) = \hat{x}(0)$$

may be replaced by the condition

$$\delta x(\hat{T}) + \frac{d\hat{x}}{dt}(\hat{T})\delta T = \delta x(0)$$

and then, since

$$\frac{d\hat{x}}{dt}(\hat{T}) = f(\hat{x}(\hat{T}),\ \hat{u}(\hat{T}))$$

$$= f(\hat{x}(0),\ \hat{u}(0))$$

we have

$$\delta x(\hat{T}) + f(\hat{x}(0),\ \hat{u}(0))\delta T = \delta x(0) \quad . \tag{3.15}$$

Using (3.14) and (3.15) in (3.12) we see that

$$\frac{dJ}{d\alpha}(x(\cdot,\alpha),\ u(\cdot,\alpha),\ T(\alpha))\big|_{\alpha=0} =$$

$$\Big(\lambda(\hat{T})^* + \frac{\partial v}{\partial x}(\hat{x}(0)) - \lambda(0)^*\Big)\delta x(0) - \Big(\lambda(\hat{T})^* f(\hat{x}(0),\ \hat{u}(0))$$

$$- \frac{1}{\hat{T}} w(\hat{x}(0),\ \hat{u}(0)) + \frac{1}{\hat{T}^2}\int_0^{\hat{T}} w(\hat{x}(t),\ \hat{u}(t))dt\Big)\delta T$$

$$+ \int_0^{\hat{T}} \Big[\frac{1}{\hat{T}}\frac{\partial w}{\partial u}(\hat{x}(t),\ \hat{u}(t)) - \lambda(t)^* H(t)\Big]\delta u(t)dt \quad . \qquad (3.16)$$

The dependence on $\delta x(0)$ is annulled by imposing the boundary condition

$$\lambda(T) - \lambda(0) + \frac{\partial v}{\partial x}(\hat{x}(0))^* = 0 \quad . \qquad (3.17)$$

At this point we have to consider two possibilities, corresponding to the two cases delineated in the above Assumption. If case (i) obtains, $\Phi(\hat{T})$ has a simple eigenvalue of 1. Then the fundamental solution matrix

$$\Psi(t) = (\Phi(t)^*)^{-1} \quad ,$$

obeying

$$\frac{d\Psi(t)}{dt} = -F(t)^*\Psi(t)$$

is such that $\Psi(\hat{T})$ has an eigenvalue equal to 1 and

$$\dot{\lambda}(t) = -F(t)^*\lambda(t)$$

has a one parameter family of $\hat{T}$-periodic solutions which we may represent in the form $\beta\tilde{\lambda}(t)$. Given a solution $\hat{\lambda}(t)$ of (3.13), (3.17)

$$\lambda(t) = \hat{\lambda}(t) + \beta\tilde{\lambda}(t)$$

also satisfies (3.13), (3.17) for every value of β. Then we may expect to be able to select β (with suitable non-degeneracy requirements) so that $\lambda(t)$ also satisfies

$$\lambda(\hat{T})^* f(\hat{x}(0), \hat{u}(0)) - \frac{1}{\hat{T}} w(\hat{x}(0), \hat{u}(0) + \frac{1}{\hat{T}^2} \int_0^{\hat{T}} w(\hat{x}(t), \hat{u}(t))dt = 0 \quad . \tag{3.18}$$

With $\lambda(t)$ so chosen, we see from (3.16) that $dJ/d\alpha = 0$ for all $\delta u(t)$ just in case

$$\lambda(t)^* H(t) - \frac{1}{\hat{T}} \frac{\partial w}{\partial u} (\hat{x}(t), \hat{u}(t)) = 0, \; t \in [0,\hat{T}] \quad . \tag{3.19}$$

If case (ii) obtains we have noted that we can find perturbed periodic solutions for which $\delta T = 0$. This also annuls the second term in (3.16) and enables us to conclude that (3.19) must hold. But, in case (ii) still, it is also possible, for each δu and α sufficiently small in (3.4), to find perturbed periodic solutions $x(t,\alpha)$ near $\hat{x}(t)$ with periodic $\hat{T} + \alpha\, \delta T$, $\delta T \neq 0$ given in advance (this is true because (cf. (3.15)) we can always solve

$$(\Phi(\hat{T}) - I)\delta x(0) + f(\hat{x}(0), \hat{u}(0))\delta T = 0$$

in case (ii)). Assuming (3.19) already established as noted, $dJ/d\alpha$ reduces to the second term in (3.16) and we conclude that condition (3.18) must, in fact, hold in this case also.

It will be observed that if $\frac{\partial v}{\partial x}(\hat{x}(0)) = 0$, (3.17) becomes

$$\lambda(\hat{T}) - \lambda(0) = 0$$

so that $\lambda(t)$ is periodic with period $\hat{T}$.

It is customary to refer to

$$\begin{aligned} &H(\lambda(t), x(t), u(t), T) \\ &\quad = \lambda(t)^* f(x(t), u(t)) - \frac{1}{T} w(x(t), u(t)) \end{aligned} \tag{3.20}$$

as the <u>Hamiltonian</u> for this optimization problem. The condition (3.19) is

$$\frac{\partial H}{\partial u} (\lambda(t), \hat{x}(t), \hat{u}(t), \hat{T}) = 0, \; t \in [0,\hat{T}] \quad . \tag{3.21}$$

We have the following result.

Proposition 3.1: $H(\lambda(t), \hat{x}(t), \hat{u}(t), \hat{T})$ is constant for $t \in [0,\hat{T}]$.

Proof. From (3.5), (3.8), (3.20) we have

$$\frac{d}{dt} H(\lambda(t), \hat{x}(t), \hat{u}(t), \hat{T}) = \dot{\lambda}(t)^* f(\hat{x}(t), \hat{u}(t))$$

$$+ \left(\lambda(t)^* F(t) - \frac{1}{\hat{T}} \frac{\partial w}{\partial x} (\hat{x}(t), \hat{u}(t))\right)\dot{\hat{x}}(t)$$

$$+ \left(\lambda(t)^* H(t) - \frac{1}{\hat{T}} \frac{\partial w}{\partial u} (\hat{x}(t), \hat{u}(t))\right)\dot{\hat{u}}(t)$$

$$= \text{(using (3.19), (3.1), (3.13))} =$$

$$[-\lambda(t)^* F(t) + \frac{1}{\hat{T}} \frac{\partial w}{\partial x} (\hat{x}(t), \hat{u}(t))]\dot{\hat{x}}(t)$$

$$+ [\lambda(t)^* F(t) - \frac{1}{\hat{T}} \frac{\partial w}{\partial x} (\hat{x}(t), \hat{u}(t))]\dot{\hat{x}}(t) = 0 \quad .$$

When $\lambda(\hat{T}) = \lambda(0)$, (3.19) will ordinarily give $\hat{u}(\hat{T}) = \hat{u}(0)$ and, since $\hat{x}(\hat{T}) = \hat{x}(0)$, (3.18) becomes

$$H(\lambda(t), \hat{x}(t), \hat{u}(t), \hat{T}) \equiv \frac{1}{\hat{T}^2} \int_0^{\hat{T}} w(\hat{x}(s), \hat{u}(s))ds, \; t \in [0,\hat{T}] \quad . \tag{3.21}$$

If we let

$$\mu(t) = \hat{T}\,\lambda(t), \; t \in [0,\hat{T}] \quad ,$$

we have, for $t \in [0,\hat{T}]$,

$$\dot{\mu}(t) = -F(t)^* \mu(t) + \frac{\partial w}{\partial x} (\hat{x}(t), \hat{u}(t)) \quad , \tag{3.22}$$

$$\mu(t)^* H(t) - \frac{\partial w}{\partial u} (\hat{x}(t), \hat{u}(t)) = 0 \quad , \tag{3.23}$$

$$\frac{d}{dt} \tilde{H}(\mu(t), \hat{x}(t), \hat{u}(t)) = 0 \quad , \tag{3.24}$$

where

$$\tilde{H}(\mu(t), x(t), u(t)) = \mu(t)^* f(x(t), u(t)) - w(x(t), u(t)) \quad . \tag{3.25}$$

Moreover, (3.20) shows that, for $t \in [0,\hat{T}]$

$$\tilde{H}(\mu(t), \hat{x}(t), \hat{u}(t)) + v(\hat{x}(0)) \equiv J(\hat{x},\hat{u},\hat{T}) \quad . \tag{3.26}$$

In this context (3.17) becomes

$$\mu(\hat{T}) - \mu(0) + \hat{T}\frac{\partial v}{\partial x}(\hat{x}(0))^* = 0 \tag{3.27}$$

and (3.18) is now

$$\mu(\hat{T})^* f(\hat{x}(0), \hat{u}(0)) - w(\hat{x}(0), \hat{u}(0) + \frac{1}{\hat{T}}\int_0^{\hat{T}} w(\hat{x}(t), \hat{u}(t))dt = 0 \quad . \tag{3.28}$$

The *necessary conditions*, modified in this way, are somewhat easier to work with because the period $\hat{T}$ occurs only in the equations (3.27), (3.28) and not in (3.22) (cf. (3.13)).

We summarize:

<u>Theorem 3.2</u>: <u>Let</u> $\hat{x}(t)$ <u>be a periodic solutions of</u> (3.1) <u>corresponding to the control</u> $\hat{u}(t)$ <u>with least positive period</u> $\hat{T}$. <u>If</u> $\mu(t)$ <u>obeys the differential equation</u> (3.13) <u>and the conditions</u> (3.27) <u>and</u> (3.28), <u>a necessary condition in ordr that</u> $\hat{x}$, $\hat{u}$, $\hat{T}$ <u>should afford a local minimum for</u> $J(x,u,T)$ <u>is</u>

$$\frac{\partial\tilde{H}}{\partial u}(\mu(t),\hat{x}(t),\hat{u}(t))$$

$$= \mu(t)^* H(t) - \frac{\partial w}{\partial u}(\hat{x}(t),\hat{u}(t)) = 0, \; t \in [0,\hat{T}] \quad . \tag{3.29}$$

<u>In this case</u>

$$J(\hat{x},\hat{u},\hat{T}) \equiv \tilde{H}(\mu,(t),\hat{x}(t),\hat{u}(t)) + v(\hat{x}(0)), \; t \in [0,\hat{T}] \quad .$$

Very little needs to be changed in the development which leads up to Theorem 3.2 in order to permit $\hat{u}$ to be a function $L^2[0,\hat{T}]$ in the case of the special class of systems (3.2). In fact, the class is selected so that with controls $u(t,\alpha)$ as in (3.4), but $\hat{u}(t)$, $\delta u(t)$ now in $L^2[0,\hat{T}]$, the resulting solution $x(t,\alpha)$ still has the form (3.10), with $\delta x(t)$ satisfying (3.11). For general functions $f(x,u)$ it is not easy to see that this is the case without making some rather technical assumptions on f to account for the possibility that, while $\|\delta u\|$ in $L^2[0,T]$ might be small, $\delta u(t)$ might still assume arbitrarily large values. The form (3.2) can be replaced by various boundedness assumptions on $\frac{\partial f}{\partial u}$ but we elect not to pursue this here. The time variable in the case of (3.2) with $\hat{u}$ in $L^2[0,\hat{T}]$ should be normalized in such a way that 0 is a Lebesgue point for $\hat{u}$, with the effect that $\frac{d\hat{x}}{dt}(0)$ is, indeed, equal to $g(\hat{x}(0) + H(\hat{x}(0))\hat{u}(0)$, so that (3.15) appropriately modified, still makes sense. (Of course $\hat{u}(t)$ turns out in nearly all cases to be smooth and periodic.)

It is not generally very easy to prove the existence of solutions of the equations constituting this set of necessary conditions. However, under the circumstances which correspond to Theorem 2.3 it is possible to prove the existence of a unique solution (in a certain sense) for small values of the parameter σ, provided we adjoin to (2.44), (2.45), (2.46) the further condition that

$$\frac{\partial w}{\partial x}(0,x(t)) \equiv 0, \quad t \in [0,T_0] \tag{3.30}$$

and the function $v(x)$ appearing in (3.3) satisfies certain requirements, about which we will have more to say below. Let us notice first of all that when $v(x) \equiv 0$ there is no possibility of a unique minimum for the cost functional (3.3) under the stipulated circumstances because translation by a fixed δT along the periodic optimal trajectory $\hat{x}(t)$, $\mu(t)$ ((3.27) shows $\mu(t)$ to be periodic when $v(x) \equiv 0$) yields a new trajectory with the same cost as well as a new set, strictly speaking, of solutions of the necessary conditions. This sort of non-uniqueness is not very interesting, of course, and it is customary and convenient to rule it out by imposing an appropriate normalizing condition on the initial state $x(0)$. Letting (cf. Section 1)

$$p = \dot{x}(0) = f(x(0),0)$$

one can require, for example, that a nearby optimal $\hat{x}(t)$ should satisfy

$$p^*(\hat{x}(0) - x(0)) = 0 \tag{3.31}$$

thereby confining the initial state $\hat{x}(0)$ to an $(n-1)$-dimensional hyperplane transverse to the nominal orbit $x(t)$. Computationally it is preferable not to enforce (3.31) as an explicit constraint but, instead, to use for $v(x)$ a function such as

$$\begin{aligned} v(x) &= (p^*(\hat{x}(0) - x(0)))^2 \\ &= (\hat{x}(0) - x(0))^* pp^*(\hat{x}(0) - x(0)) \quad . \end{aligned} \tag{3.32}$$

In obtaining the least possible cost for (3.3) one automatically obtains the periodic $\hat{x}(t)$ which satisfies (3.31) and, in the end, $\frac{\partial v}{\partial x}(\hat{x}(0)) = 0$ so that (3.27) applies to show $\mu(t)$ to be periodic.

Provided a correctly constructed normalizing $v(x)$ is used as indicated above, it is possible to use the implicit function theorem to construct $\hat{x}(t)$, $\mu(t)$, near $x(t)$, 0, satisfying the necessary conditions set forth in Theorem 3.2. One takes $\hat{x}(0)$, $\mu(0)$ to lie in a small neighborhood of $x(0)$, 0. Then, via solution of the differential equations (3.1), (3.13) with $\hat{u}(t)$ determined by (3.19), the vectors

$$\hat{x}(\hat{T}) - \hat{x}(0) \quad , \tag{3.33}$$

$$\mu(\hat{T}) - \mu(0) + \hat{T}\frac{\partial v}{\partial x}(\hat{x}(0)) \tag{3.34}$$

and the scalar quantity (cf. (3.28))

$$H(\mu(0),\hat{x}(0),\hat{u}(0)) + v(\hat{x}(0)) - \frac{1}{\hat{T}}\int_0^{\hat{T}} w(\hat{x}(t),\hat{u}(t))dt \tag{3.35}$$

are seen to constitute a $(2n+1)$ dimensional vector function of $\hat{x}(0)$, $\mu(0)$, T and any relevant parameters, such as σ of Theorem 2.3 or the parameter μ, as in the discussion of the alternate problem following Theorem 2.3. The Jacobian of (3.33), (3.34), (3.35) with respect to $\hat{x}(0)$, $\mu(0)$ and $\hat{T}$ can be computed with the aid of the variational equations and under appropriate circumstances (e.g. the cost functional has the form discussed in Theorem 2.3, (3.30) holds and $v(x)$ is defined as in (3.32)) one can show that this Jacobian is nonsingular when $\sigma = 0$ (or, depending on the problem, when $\mu = 0$) and unique $\hat{x}(0)$, $\mu(0)$, $\hat{T}$ can then be inferred for σ (or μ) sufficiently small. We will not go into detail on this as it will appear elsewhere and would lead us much too far astray here.

The computational version of the above is the simple Newton's method for solving the equations obtained by setting (3.33), (3.34), (3.35) equal to zero. Implementation requires numerical integration of the variational equations over an interval $[0,T_k]$, where T_k is the latest approximation to the period $\hat{T}$ in order to be able to compute the Jacobian of (3.33), (3.34), (3.35) with respect to the current initial conditions for x_k, μ_k and the period. Just as noted above, it can be established that this procedure is locally convergent in certain cases. However, this method, while theoretically useful and appealing in its simplicity, suffers from a number of disadvantages as a computational procedure. The necessity of integrating the variational equations to obtain the Jacobian matrix is one such disadvantage to begin with, but more serious is the frequent numerical instability of the procedure. In most cases the periodic solution $x(t)$, corresponding to $u(t) = 0$, that one begins with, is asymptotically stable. The corresponding variational system is a time varying periodic system whose characteristic exponents have negative real parts. But the dual variational system for the adjoint variable μ (or λ) has characteristic exponents which are the negatives of those of the variational system for $x(t)$. Thus the variational system for μ (or λ) has solutions which may grow very rapidly. This means that it is often very difficult to accurately compute the part of the Jacobian corresponding to the equation (3.34). This has been a serious limiting factor in our computational experience thus far.

While theoretically less rapidly convergent, the steepest descent method offers much in the way of improved performance from the standpoint of numerical stability, primarily because the state equations and adjoint equations are integrated in opposite directions to take advantage of their differing stability characteristics. We will describe briefly, one step of such a procedure.

We shall suppose that $x_k(t)$, $u_k(t)$ form a periodic trajectory - control pair for (3.1) with period T_k. As we do not require, ab initio, any continuity for u_k, the periodicity refers to the trajectory $x_k(t)$. With $x(t) = x_k(t)$ available, one proceeds to compute a solution $\mu_k(t)$ of (3.13). Corresponding to the two cases admitted under the Assumption above, one follows two routes. If case (i) applies, $\mu_k(t)$ is determined so that it also satisfies (3.28). Then setting

$$\delta u_k(t) = H(t)^* \mu_k(t) - \frac{\partial w}{\partial u}(\hat{x}(t),\hat{u}(t))^* \qquad (3.36)$$

gives (cf. (3.16)) $dJ/d\alpha < 0$ (if (3.23) does not already hold for x_k, μ_k). For small positive α we let

$$u_{k+1}(t) = u_k(t) + \alpha\, \delta u_k(t), \quad t \in [0,T_k] \quad .$$

Extension of $\delta u_k(t)$ outside $[0,T_k]$, where required, can be accomplished from (3.36) and we may assume this has already been done for $u_k(t)$. Then we determine $\delta x_k(0)$ and δT_k so that $x_{k+1}(t)$, the solution of (3.1) with initial state $x_k(0) + \delta x_k(0)$ and $u(t) \equiv u_{k+1}(t)$ satisfies $x_{k+1}(T_k + \delta T_k) = x_{k+1}(T_{k+1}) = x_{k+1}(0)$. if $\alpha > 0$ is sufficiently small one can show that the cost associated with x_{k+1}, u_{k+1} will be smaller then that associated with x_k, u_k. If case (ii) applies μ_k is only required to satisfy (3.27), δu_k is selected as in (3.36), and we let $\delta T_k = \alpha$ times the quantity on the left hand side of (3.28); we will again see from (3.16) that $dJ/d\alpha < 0$. This defines the new period $t_{k+1} = T_k + \delta T_k$ and we proceed, as we may in case (ii), to find x_{k+1}, corresponding to the control u_{k+1}, having precisely the period T_{k+1}. Again the cost associated with x_{k+1}, u_{k+1} will be reduced, as compared with x_k, u_k, if $\alpha > 0$ is sufficiently small.

The above procedure is continued until the necessary conditions are satisfied. The integrations necessary to determine $u_k(t)$ are always carried out in reverse time (as compared with those for $x_k(t)$) for the sake of stability. The above description is very rough and a number of refinements remain to be delineated; for example, how do we treat the situation which obtains when case (i) is very nearly, but not quite, true. This is not the place to treat such questions.

While case (ii) may be considered generic for our problem, case (i) does arise in significant instances; e.g. with a cost functional (2.47), satisfying (3.30), the initial step away from the uncontrolled solution must be made under case (i) circumstances.

<u>An Example</u>. Initial computational experiments have been carried out with the second order scalar system

$$\ddot{x} + (-\theta + .1(\dot{x})^2\dot{x} + x = u \qquad (3.37)$$

with the parameter θ in the range $0 < \theta < .6$ with the objective of amplitude suppression. The uncontrolled system has a self-excited oscillation with amplitude $A(\theta)$ shown by the solid curve $A(\theta)$ in Figure 3.2, the whole orbit for $\theta = .6$ being shown by the closed solid curve in Figure 3.1.

Initially we worked with a cost functional of the form

$$\frac{1}{T} \int_0^T [\sigma(x(t)^2 + (\dot{x}(t))^2) + u(t)^2]dt \quad , \tag{3.38}$$

first computing the uncontrolled orbit, which is optimal for $\sigma = 0$ and then increasing σ in small increments. The computational method used was the Newton method described earlier. The cost functional (3.37) has the form discussed in Theorem 2.3. It proved unsatisfactory, however. In a very narrow range of positive values a controlled periodic solution of smaller amplitude was obtained but it "popped", like a bubble, for $\sigma \approx .023$. The word "popped" is used advisedly because we believe the disappearance of the solution near this value of σ is indeed due to mathematical causes similar to those which are operative in the bursting of a bubble or the stretching of a soap film; the stability of the initial uncontrolled solution manifests itself initially - a finite amount of control energy is required just to move away from this initial solution. Beyond a certain point, however, a minimal cost solution of positive amplitude no longer exists - the origin is the only (degenerate) solution. We are fairly well satisfied that the optimal solution does, in fact, disappear in "mid-air" rather than collapse continuously toward the origin - not only by numerical results but also by related theoretical considerations which we shall not go into here.

Better success was achieved by following the alternate schema suggested in the material which follows Theorem 2.3 in Section 2. We used a fixed cost functional of the form

$$\frac{1}{T}\int_0^T \left[\frac{1}{4}\left(R(t)^2 - \alpha^2\right)^2 + (u(t))^2\right]dt \tag{3.39}$$

where

$$R(t)^2 = (x(t))^2 + (\dot{x}(t))^2$$

and α is the average amplitude of the uncontrolled solution when $\theta = .4$. The condition $W(\breve{x}(t)) \equiv 0$ (cf. (2.55) pp) is not quite satisfied in this case but the uncontrolled orbit is so nearly circular that the discrepancies have negligible effect. When $\theta = .4$ there is no noticeable modifications of the orbit but, as θ is increased the effect of the optimal control policy is more and more pronounced, producing the modified amplitude curve $A_c(\theta)$ shown by the dashed curve in Figure 3.2 and, for $\theta = .6$, the modified orbit shown as a dashed, closed curve in Figure 3.1.

The extreme "fragility" of the Newton method became apparent in these computations; whether the parameter was σ in (3.38) or θ in (3.37) in connection with (3.39), it was necessary to change it in extremely small increments to avoid numerical "blow-up" - primarily due to the instability of the adjoint system as one integrates in the positive t direction.

In future work we hope to use the steepest descent method outlined here to compute optimal controls and modified orbits for a system representative of the wing - air system described in Section 1 and shown in Figure 1.2. The difficulties already noted for (3.38) indicate that the use of the Newton method would be almost hopeless in this case.

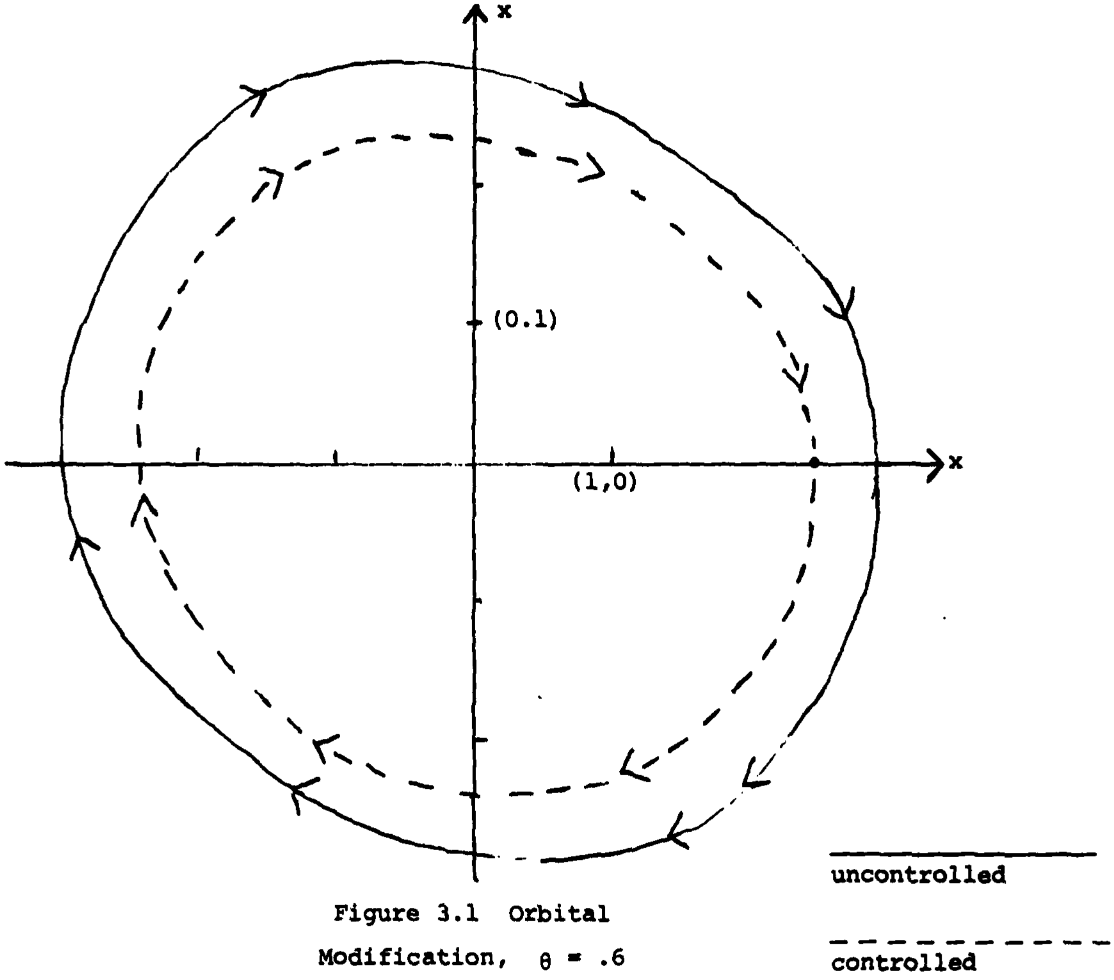

Figure 3.1 Orbital Modification, θ = .6

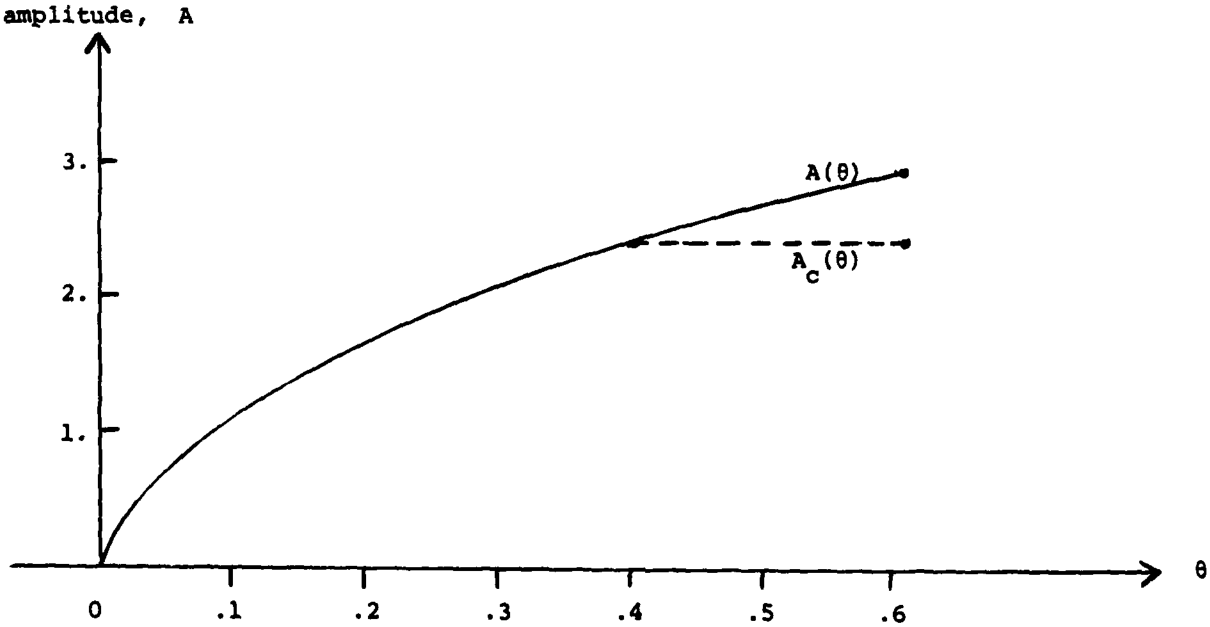

Figure 3.2 Controlled and Uncontrolled Amplitude vs Parameter, θ.

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Note: [I] and [K] contain long list of additional references on periodic optimal control and [A] has an extensive literature on the Hopf bifurcation.

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We consider a controlled system

$$\dot{x} = f(x,u)$$

exhibiting a self excited periodic solution $x(t)$ for $u(t) \equiv 0$ and consider the question of modification of this orbit to a controlled periodic solution $\hat{x}(t)$ corresponding to a control $\hat{u}(t)$ chosen so as to minimize a cost functional of the form

$$\frac{1}{\hat{T}} \int_0^{\hat{T}} w(\hat{x}(t), \hat{u}(t))dt \quad ,$$

where $\hat{T}$ is the period of the controlled periodic solution. Some relevant applications are cited.

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